Random Graphs and Limits Semester I 2020/21

Final Exam **Date : Dec 24th, 2020.** **Time : 9.30 - 13.30.**

Maximum Points : 100. **Attempt as many questions as you can.**

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1. (20 points) Let G_n be a random graph on the vertex set $V(G_n) = \{1, 2, \ldots, n\}$ vertices such that its edge set $E(G_n)$ is given by

$$\{i,j\} \in E(G_n) \begin{cases} \text{with probability } \frac{1}{10} & \text{if } \min\{U_i, U_j\} < \frac{1}{3} \\ \text{with probability } \frac{1}{100} & \text{if } \max\{U_i, U_j\} > \frac{1}{3} \text{ and } \min\{U_i, U_j\} < \frac{1}{3} \\ \text{with probability } \frac{1}{25} & \text{if } \min\{U_i, U_j\} > \frac{1}{3} \end{cases}$$

where $\{U_n : n \geq 1\}$ be a sequence of i.i.d. Uniform $[0,1]$ random variables
Suppose t_n^2, t_n^3, t_n^4 denote the edge density, triangle density and the 4-cycle density in G_n.

(a) Show that for any $\epsilon > 0$, there is a $c > 0$ and deterministic sequence $A_n, b_n > 0$ such that

$$P(\mid t_n^i - A_n \mid > \epsilon) \leq \exp(-\frac{c\epsilon^2}{b_n}),$$

for $i \in \{1, 2, 3\}$.

(b) Decide if the sequences converge and if they do then identify the limit.

2. (10 points)Let G_n be a graph with vertex set $\{1, 2, \ldots, n\}$. Let $\hat{G_n}$ be the random labelled graph obtained by a random re-labelling of the vertices of G. Show that for any $F \in \mathcal{L}_k$,

$$\mathbb{E}(t_{\text{ind}}(F, G_n) \leq \mathbb{P}(\hat{G}_n|_{[k]} = F) \leq \mathbb{E}(t_{\text{ind}}(F, G_n) + \mathbb{P}(v(G_n) \leq k)$$

3. (20 points) Let H be an exchangeable random infinite graph in $\mathcal{L}_\infty$.

(a) Suppose the distribution of H invariant under every finite permutations, (i.e., permutations $\sigma : \mathbb{N} \to \mathbb{N}$ that satisfy $\sigma(i) = i$ for all sufficiently large i), then show that H is exchangeable.

(b) Show that the following are equivalent

i. H is exchangeable

ii. $H|_{[k]}$ has a distribution invariant under all permutations of $[k]$ for all $k \geq 1$.

4. Define the n-hypercube graph as follows : $V_n = \{0,1\}^n$ is the vertex set and edge set is $E_n = \{(v, w) : v - w = \overset{+}{-}\, e_k \text{for some } 1 \leq k \leq n\}$ i.e., (v, w) is an edge if they differ exactly at one co-ordinate. Let $H(n, p)$ denote the random graph such that each edge in E_n is chosen with probability $p \in [0, 1]$ independently of each other. Let $O := (0, \ldots, 0) \in V_n$ for all $n \geq 1$. For any $v \in V_k$, we set $v(n) := (v, 0, \ldots, 0) \in V_n$ for all $n \geq k$. Set $p_n = \min\{c/n, 1\}$ for all $n \geq 1$ where $c \in (0, \infty)$ and $H_n := H(n, p_n)$.

(a) (10 points) Let $v_1, \ldots, v_l \in V_k$ for some $l \leq 2^k$ fixed. For $n \geq k$, define $D_i(n) := deg_{H_n}(v_i(n)), 1 \leq i \leq l$ i.e., the degrees of $v_i(n)$ in H_n. Compute the asymptotic distribution of the random vector $(D_1(n), \ldots, D_l(n))$ as $n \to \infty$.

(b) (10 points) Let C_O be the component of O in H_n and C_{max} be the largest component (with ties broken arbitarily) in H_n. If $c < 1$, show that there exists a positive constant a_c such that for all $k \geq 1$ and n large enough,

$$\mathbb{P}(|C(O)| > k) \leq e^{-a_c k},$$

and further show that

$$2^{-n}|C_{max}| \xrightarrow{p} 0.$$

(c) (10 points) Let $c > 1$. Show that for any $K > 0$,

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}(d_{H_n}(o_n^{(1)}, o_n^{(2)}) \leq \frac{n \log 2}{\log c} - K) = 0,$$

where $o_n^{(1)}, o_n^{(2)}$ are independent uniformly chosen vertices in V_n.

5. Let $K_{n,n}$ be the complete bi-partite graph on $2n$ vertices defined as follows : $V_n = [2n]$ is the vertex set and $E_n = \{(i,j) : 1 \leq i \leq n, n+1 \leq j \leq 2n\}$ is the edge-set. We define $G_n := G(n, p_n)$ the bi-partite random graph by choosing each edge in E_n independently of each other with probability $p_n = \min\{\lambda/n, 1\} \in [0,1]$. Show the following :

   (a) (10 points) Let C_k be the k-cycle and T_k be a tree on k vertices. As defined in the class, let $X(C_k, G_n)$ and $X(T_k, G_n)$ be the number of copies of C_k and T_k in G_n respectively. Compute the limits of $\mathbb{E}[X(C_k, G_n)]$ and $n^{-1}\mathbb{E}[X(T_k, G_n)]$ as $n \to \infty$.

   (b) (10 points) Let $\mathcal{H}_n(1) = (\xi_1, \ldots, \xi_m)$ be the history of the breadth-first exploration in G_n starting at 1 upto step m. Show that

$$\mathbb{P}(\xi_1 = x_1, \ldots, \xi_k = x_m) = \prod_{i=1}^{m} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!},$$

   for any feasible sequence $(x_1, \ldots, x_m)$.

   (c) (10 points) What is the limit of G_n under local weak convergence ? Give a sketch of proof to justify the limit i.e., outline the main steps involved in the proof and provide some details on how you will prove the steps. You are not required to give a complete proof of the steps.

6. (10 points)

   (a) Let (G, o) be a unimodular random rooted graph. Suppose that $\phi : \mathcal{G}_{**} \to \mathbb{R}_+$ be a measurable function on the space of doubly rooted graphs. Set $w'(u,v) = \phi(G, u, v)$ for $(u,v) \in E(G)$. Show that (G, w, o) is a unimodular random weighted rooted graph.

   (b) Let (G, w, o) be a unimodular random weighted rooted graph with $w \in \mathbb{W}$, a Polish space. Let $B \subset \mathbb{W}$ be a Borel subset. Define a new weighted graph (G', w') where $V' = V$ and edge-set $E' = \{(u,v) : w(u,v) \in B, w(v,u) \in B\}$ and $w'(u,v) = w(u,v)$ for $(u,v) \in E'$. Show that (G', w', o) is a unimodular random weighted rooted graph.